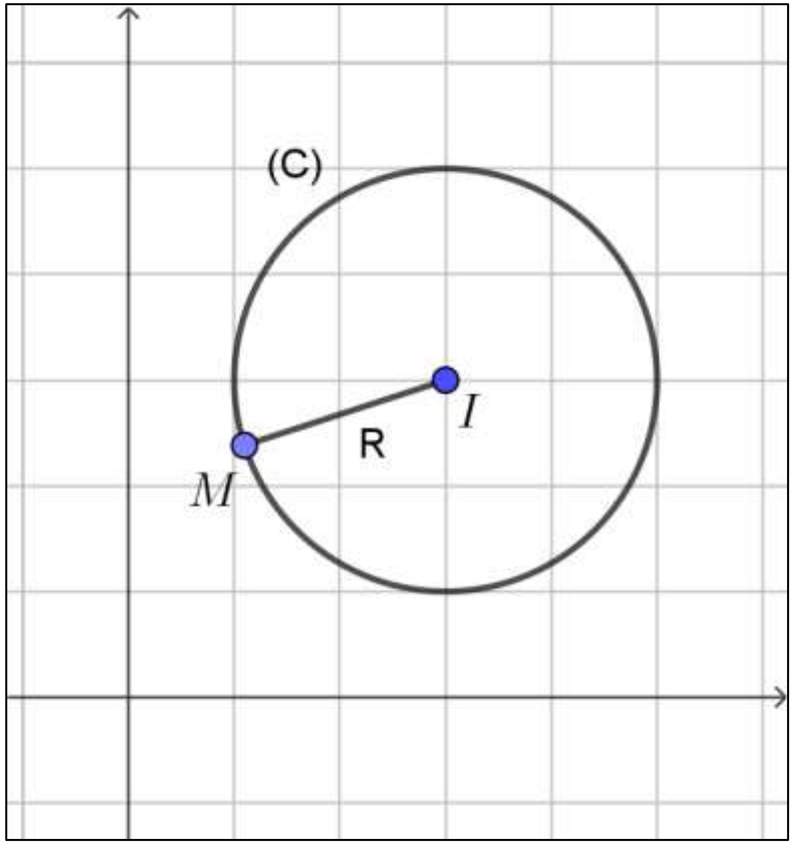
Equations of a Circle

Circle defined by its center and radius

Definition

The circle of center I(a; b) and radius R is the set of all the points of the plane whose distance to I is equal to R.



Let M(x; y) a point of the plane such that IM=R.

$$IM^2 = R^2$$

$$(x_M - x_I)^2 + (y_M - y_I)^2 = R^2$$

$$(x - a)^2 + (y - b)^2 = R^2$$

Standard equation of a circle

Circle defined by its center and radius

Application ❶

Find the equation of the circle of center I and radius R in each case:

❶ I(1;2) and R = 1

$$(x - x_I)^2 + (y - y_I)^2 = R^2$$
$$(x - 1)^2 + (y - 2)^2 = 1$$

❷ I(-1;3) and R = 2

$$(x - x_I)^2 + (y - y_I)^2 = R^2$$
$$(x + 1)^2 + (y - 3)^2 = 4$$

❸ I(-1;-2) and R = 1

$$(x - x_I)^2 + (y - y_I)^2 = R^2$$
$$(x + 1)^2 + (y + 2)^2 = 1$$

Circle defined by its center and radius

Application 2

Find the center I and the radius R of the circle in each case.

1. $(x-2)^2+(y+3)^2=25$

   Center: I(2;-3)

   Radius: R = 5

2. $(x+2)^2+y^2-36=0$ $(x+2)^2+y^2=36$

   Center: I(-2;0)

   Radius: R = 6

3. $x^2+(3-y)^2=64$ $x^2+(y-3)^2=64$

   Center: I(0;3)

   Radius: R = 8

Circle defined by its center and radius

Standard equation of a circle:

$$(x-a)^2+(y-b)^2=R^2$$

$$\Updownarrow$$

c

$$x^2+y^2-2ax-2by+a^2+b^2-R^2=0$$

$$c=a^2+b^2-R^2$$

So

$$R^2=a^2+b^2-c$$

Circle defined by its center and radius

Standard form of the equation

$(x - a)^2 + (y - b)^2 = R^2$

Expanded form of the equation

$x^2 + y^2 - 2ax - 2by + c = 0$

Where $R^2 = a^2 + b^2 - c$

Circle defined by its center and radius

Application ❸

Find the center I and the radius R of the circle in each case.

❶ $x^2 + y^2 - 4x + 6y - 12 = 0$

Expanded form of the equation
$x^2 + y^2 - 2ax - 2by + c = 0$
Where $R^2 = a^2 + b^2 - c$

$-2a = -4$; $a = -\frac{4}{-2} = 2$

$-2b = 6$; $b = \frac{6}{-2} = -3$

$R^2 = a^2 + b^2 - c = 4 + 9 - (-12) = 25$

So circle of center I(2;-3) and radius R = 5

Circle defined by its center and radius

Application ❸

Find the center I and the radius R of the circle in each case.

❷ $x^2 + y^2 + 2x + 4y - 1 = 0$

$-2a = 2; \quad a = \frac{2}{-2} = -1$

$-2b = 4 \ ; \quad b = \frac{4}{-2} = -2$

$R^2 = a^2 + b^2 - c = 1 + 4 - (-1) = 6$

So circle of center I(-1;-2) and radius R = $\sqrt{6}$

Expanded form of the equation

$x^2 + y^2 - 2ax - 2by + c = 0$

Where $R^2 = a^2 + b^2 - c$

Circle defined by its center and radius

Application ❹

Determine the set E defined by:

❶ $x^2+y^2+4x-2y+4=0$

$-2a=4$; $a=-2$

$-2b=-2$; $b=1$

$R^2=a^2+b^2-c=4+1-4=1>0$

So circle of center I(-2;1) and radius R = 1

Circle defined by its center and radius

Application ❹

Determine the set E defined by:

❷ $x^2 + y^2 - 3x + 5y + 10 = 0$

$-2a = -3\ ;\ \ a = \frac{3}{2}$

$-2b = 5\ ;\ \ b = -\frac{5}{2}$

$R^2 = a^2 + b^2 - c = \frac{9}{4} + \frac{25}{4} - 10 = \frac{34}{4} - 10 = -\frac{6}{4} = -\frac{3}{2} < 0$

So this is empty set

Circle defined by its center and radius

Application ❹

Determine the set E defined by:

❸ $x^2 + y^2 - 2x + y = 0$

$-2a = -2 \; ; \; a = 1$

$-2b = 1 \; ; \; b = -\frac{1}{2}$

$R^2 = a^2 + b^2 - c = 1 + \frac{1}{4} - 0 = \frac{5}{4}$

So circle of center I$(1;-\frac{1}{2})$ and radius $\frac{\sqrt{5}}{2}$.

Circle defined by its center and radius

Application ❹

Determine the set E defined by:

❹ $3x^2 + 3y^2 - 6x + 9y + \frac{13}{4} = 0$

$x^2 + y^2 - 2x + 3y + \frac{13}{12} = 0$

$-2a = -2$; $a = 1$

$-2b = 3$; $b = -\frac{3}{2}$

$R^2 = a^2 + b^2 - c = 1 + \frac{9}{4} - \frac{13}{4} = 0$

So the set is {I}

Circle defined by its center and radius

Application 4

Determine the set E defined by:

❹ $3x^2 + 3y^2 - 6x + 9y + \frac{13}{4} = 0$

$x^2 + y^2 - 2x + 3y + \frac{13}{12} = 0$

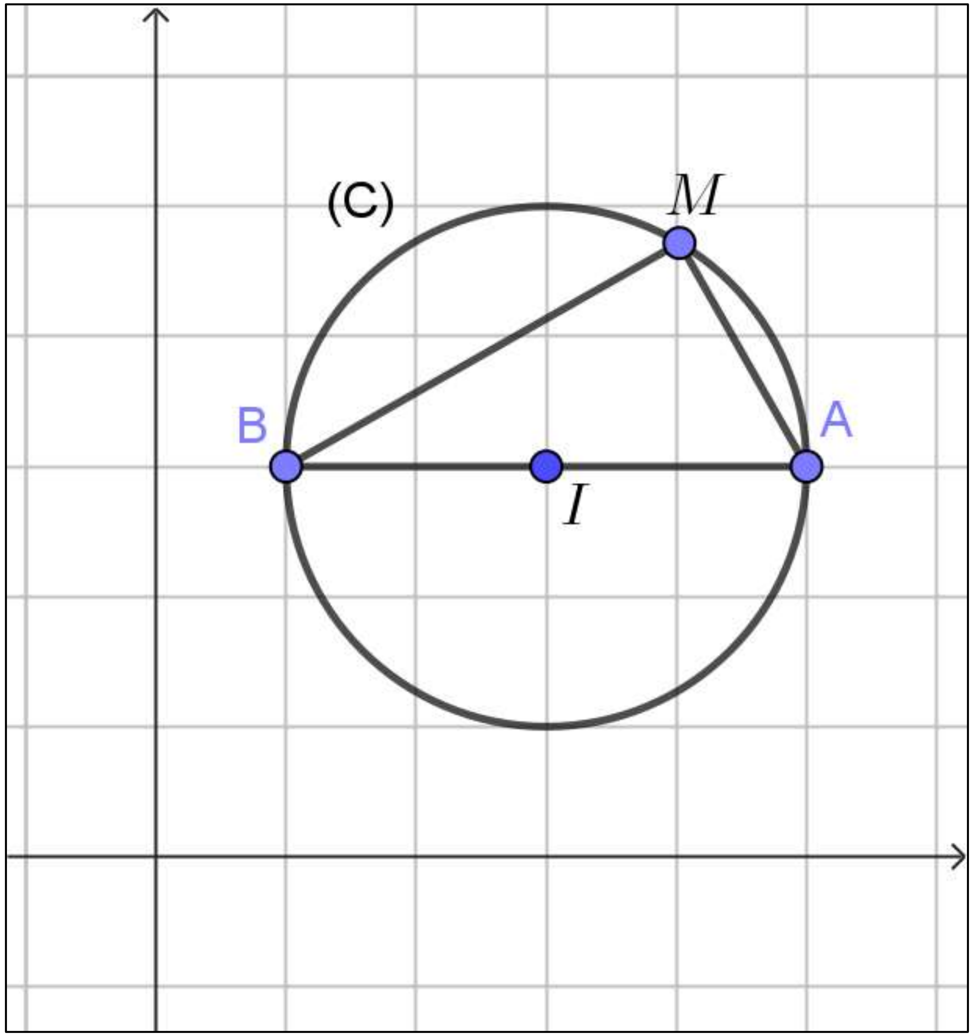
$-2a = -2 \; ; \quad a = 1$

$-2b = 3 \; ; \; b = -\frac{3}{2}$

$R^2 = a^2 + b^2 - c = 1 + \frac{9}{4} - \frac{13}{4} = 0$

So the set is {I}

Circle defined by its diameter



Let M$(x; y)$ a point of the plane.

$\widehat{BMA}$ is a right triangle at M.

$\overrightarrow{BM}.\overrightarrow{AM} = 0$

$xx' + yy' = 0$

$(x - x_B)(x - x_A) + (y - y_B)(y - y_A) = 0$

Circle defined by its center and radius

Application 5

Find the equation of the circle of diameter [AB] where A(1;2) and B(-1;3).

$(x - x_A)(x - x_B) + (y - y_A)(y - y_B) = 0$

$(x - 1)(x + 1) + (y - 2)(y - 3) = 0$

$x^2 - 1 + y^2 - 5y + 6 = 0$

$x^2 + y^2 - 5y + 5 = 0$

Power of a point with respect to a circle

Definition

Consider the circle C(I;R) and a point A of the plane.
The power of A with respect to (C) is $P(A) = AI^2 - R^2$

$$
\begin{aligned}
P(A) = AI^2 - R^2 &= (x_A - x_I)^2 + (y_A - y_I)^2 - R^2 \\
&= (x_A - a)^2 + (y_A - b)^2 - R^2 \\
&= x_A^2 - 2ax_A + a^2 + y_A^2 - 2by_A + b^2 - R^2 \\
&= x_A^2 + y_A^2 - 2ax_A - 2by_A + c
\end{aligned}
$$

Power of a point with respect to a circle

Definition

Consider the circle C(I;R) and a point A of the plane.
The power of A with respect to (C) is $P(A) = AI^2 - R^2$

Example 1:

Consider the circle (C) of equation $(x-3)^2+(y-4)^2=5$ and the point A(1;-1).

$$P(A) = (x_A-3)^2+(y_A-4)^2-5$$
$$= (1-3)^2+(-1-4)^2-5$$
$$= 4+25-5 = 24$$

Power of a point with respect to a circle

Definition

Consider the circle C(I;R) and a point A of the plane.
The power of A with respect to (C) is $P(A) = AI^2 - R^2$

Example 2:

Consider the circle (C) of equation $x^2 + 3x + y^2 - 1 = 0$ and the point A(2;0).

$P(A) = x_A^2 + 3x_A + y_A^2 - 1 = 4 + 6 + 0 - 1 = 9$

Power of a point with respect to a circle

$P(A) > 0$

A is at the exterior of the circle.

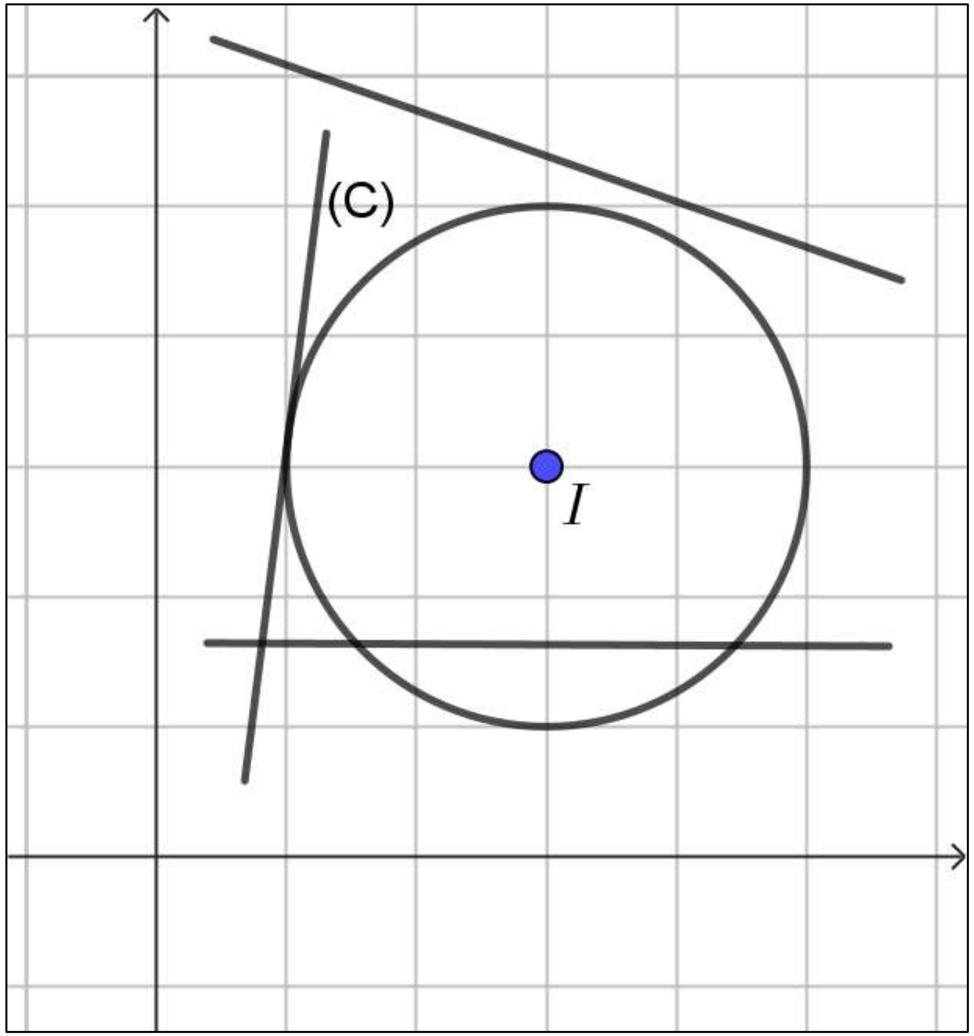
$P(A) < 0$

A is at the interior of the circle

$P(A) = 0$

A is on the circle

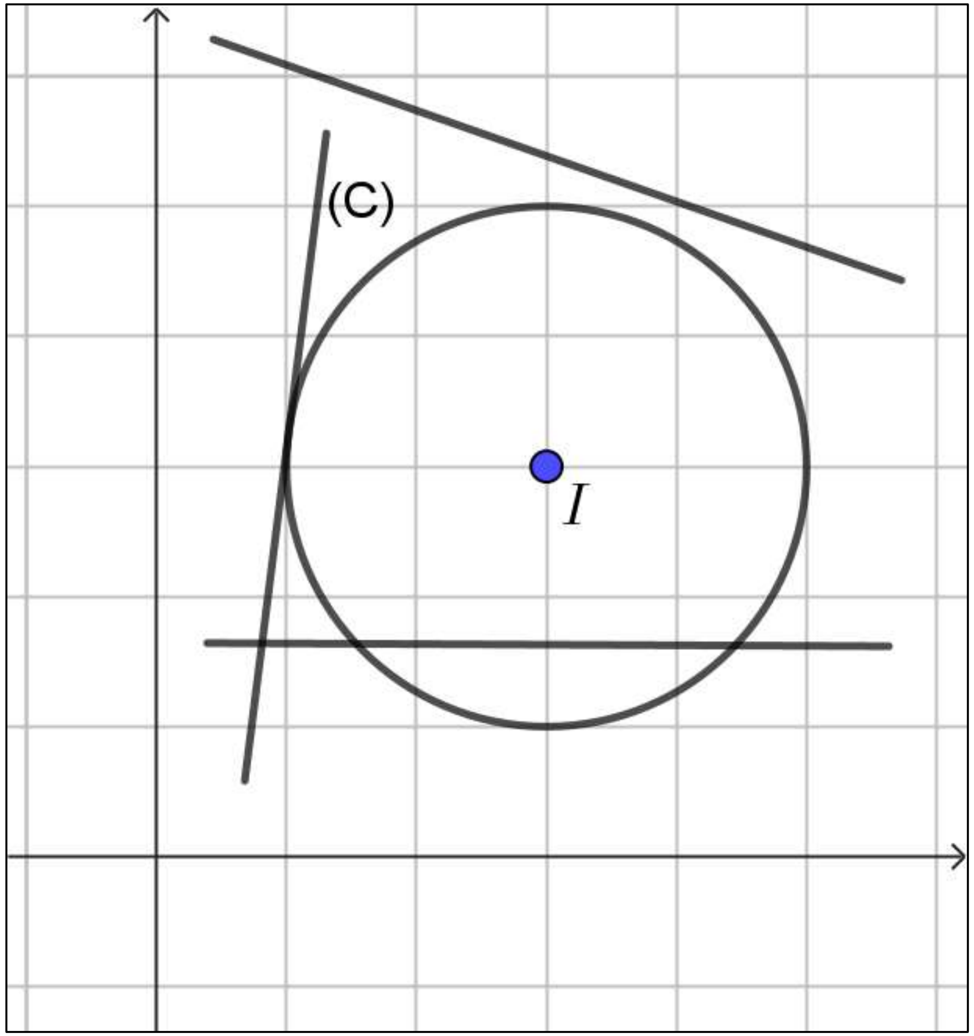
Relative position of a line and a circle



3 relative positions:

- Line at the exterior of (C) (no common points).
- Line tangent to (C) (only1 common point).
- Secant Line. (two common points).

Relative position of a line and a circle



How to determine the relative position between a line and a circle?

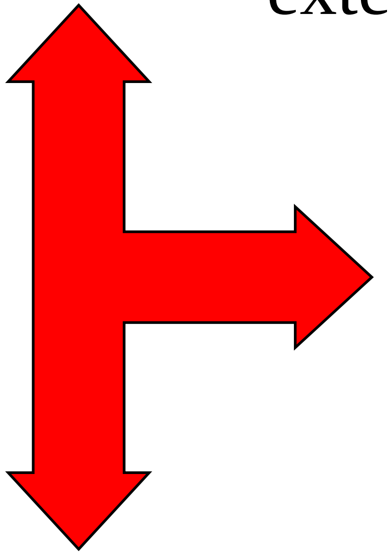
DISTANCE from the center of (C) to the line

Relative position of a line and a circle

Consider the circle C(I;R) and the line (d) of equation $ux + vy + w = 0$

$$d(I;(d)) = \frac{|ux_I + vy_I + w|}{u^2 + v^2}$$

d > R (d) is at the exterior of (C)



d = R (d) is tangent to (C)

d < R

(d) is a secant

Relative position of a line and a circle

Application 6

Consider the circle (C) of equation: $(x-1)^2+(y-1)^2=1$ and the line (d): $2x-y=0$. Show that (d) cuts the circle (C) in two points and find their coordinates.

The radius is R = 1 and I(1;1)

$$d\big(I;(d)\big)=\frac{|2x_I-y_I|}{u^2+v^2}$$

$$=\frac{|2(1)-1|}{2^2+1^2}=\frac{1}{5}<1$$

So the line (d) cuts (C) in two points.

Relative position of a line and a circle

Application 6

Consider the circle (C) of equation: $(x-1)^2+(y-1)^2=1$ and the line (d): $2x-y=0$. Show that (d) cuts the circle (C) in two points and find their coordinates.

The coordinates of the points of intersections verify the system:

$$\begin{cases} 2x-y=0 \\ (x-1)^2+(y-1)^2=1 \end{cases}$$

$y=2x$ so substitute in the equation of (C).

$(x-1)^2+(2x-1)^2=1$

$x^2-2x+1+4x^2-4x+1-1=0$ so $5x^2-6x+1=0$

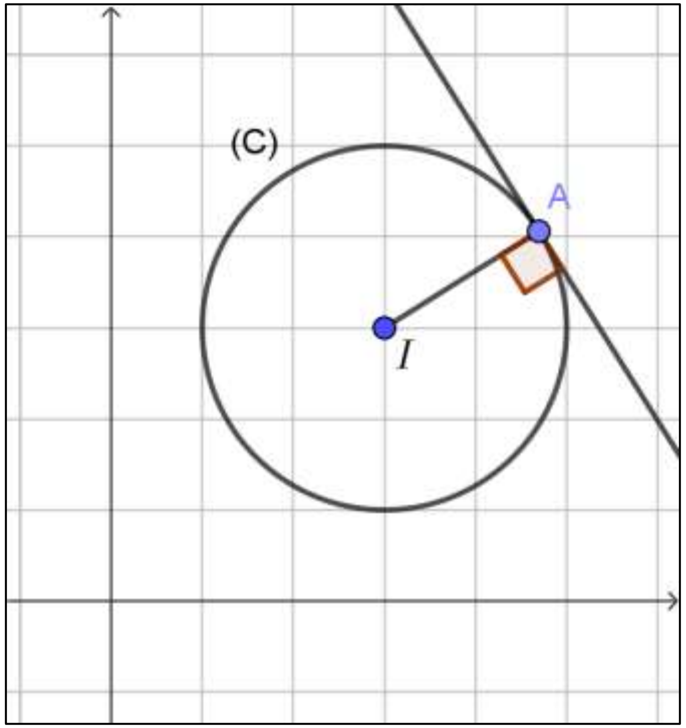
The solutions are $x=1$; $x=\frac{1}{5}$

$y=2$; $y=\frac{2}{5}$

Tangent to a circle (C) from a point A

A belongs to (C)

A is at the exterior of (C)



Equation of the tangent (D) to the circle (C) at A.

Let M(x; y) belongs to (D).

$\overrightarrow{AM}.\overrightarrow{AI} = 0$

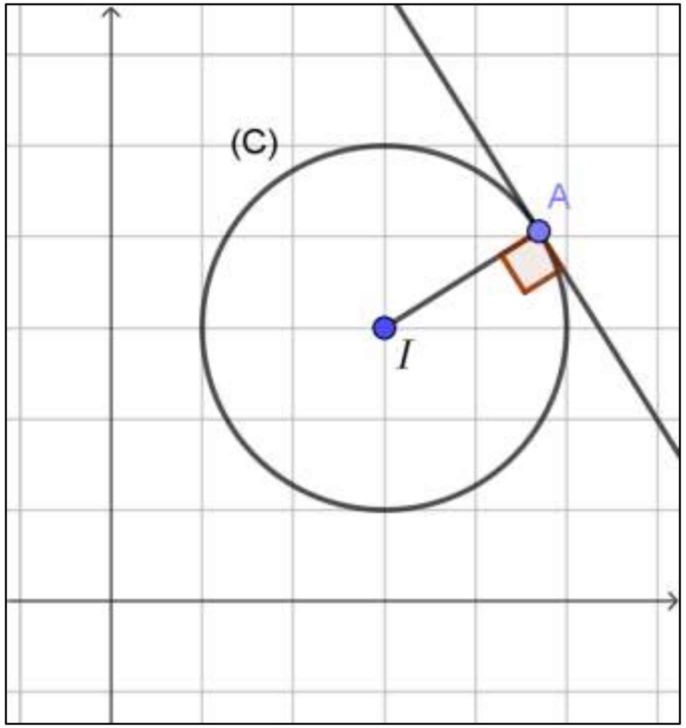
$xx' + yy' = 0$

$(x_M - x_A)(x_I - x_A) + (y_M - y_A)(y_I - y_A) = 0$

Tangent to a circle (C) from a point A

A belongs to (C)

A is at the exterior of (C)



Example: (C): $(x-1)^2 + (y+2)^2 = 4$ and A(-3;1)

Let M($x; y$) belongs to (D).

$$\overrightarrow{AM}.\overrightarrow{AI} = 0$$

$$xx' + yy' = 0$$

$$(x_M - x_A)(x_I - x_A) + (y_M - y_A)(y_I - y_A) = 0$$

$$(x+3)(1+3) + (y-1)(-2-1) = 0$$

$$4(x+3) - 3(y-1) = 0$$

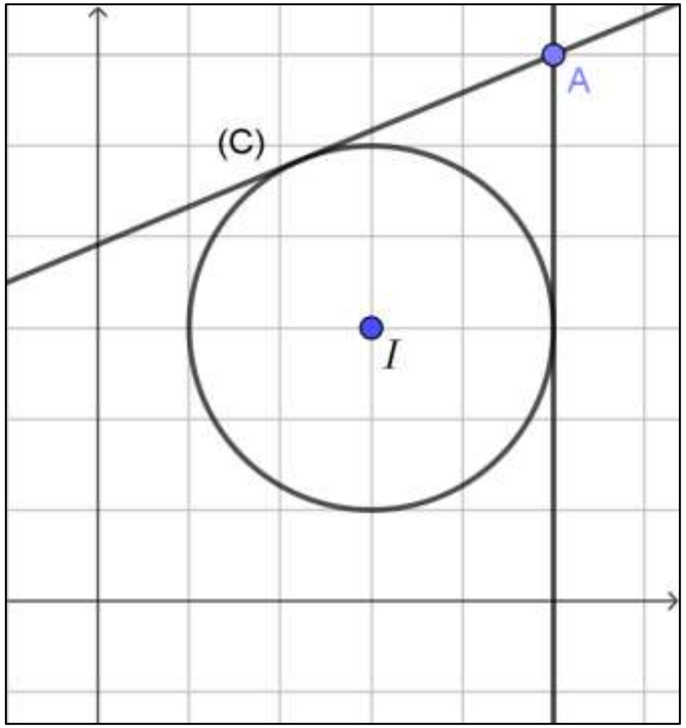
$$4x + 12 - 3y + 3 = 0$$

$$4x - 3y + 15 = 0$$

Tangent to a circle (C) from a point A

A belongs to (C)

A is at the exterior of (C)



Equation of the tangents issued from A to the circle (C).

The equation of the line (D) passing through A is $y = mx + b$; $mx + b - y = 0$

$$d\big(I;(D)\big) = R$$

Tangent to a circle (C) from a point A

Example: (C): $(x-1)^2+(y+2)^2=1$ and A(3;2)

The equation of the line that passes through A is: $y=mx+b$

$y_A=mx_A+b$; $2=3m+b$

so $b=2-3m$

Then: $y=mx+2-3m$;

$mx-y+2-3m=0$

$d(I;(D))=\frac{|m+2+2-3m|}{m^2+1}=1$; $\frac{|-2m+4|}{m^2+1}=1$

$|-2m+4|=m^2+1$

$-2m+4=m^2+1$ or $-2m+4=-m^2-1$

$m^2+2m-3=0$ $\quad$ $-m^2+2m-5=0$

$m=1$ or $m=-3$ $\quad$ no solution

So the equations of the two tangents are $y=x-1$; $y=-3x+11$